

**The Potential for Using Geomagnetic Data
in Ocean and Climate Studies. II:
Electrodynamics in a Rotating Frame
of Reference**

**Robert H. Tyler and Lawrence A. Mysak
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Abstract

In a rotating reference frame, Maxwell's equations will not, in general, retain their inertial-frame form. More importantly, the constitutive relationships will depend on both the motion of the reference frame as well as the relative motion of the medium in this reference frame.

In the limit of slow rotational velocities, the electrodynamic equations in the rotating reference frame are given for the cases of a medium stationary in the rotating frame, and a medium with a general velocity relative to the rotating frame. The last case is considered as a necessary formalism before large-scale electromagnetic induction processes in the ocean can be correctly modelled.

After time and space scaling, a governing induction equation in one vector variable (\mathbf{b}) is derived and written out explicitly in various coordinate systems.

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1 List of Symbols

B magnetic flux density (T)

H magnetic field strength (A/m)

J electric current density (A/m²)

D electric flux density (C/m²)

ρ_e electric volume charge density (C/m³)

E electric field strength (V/m)

$\mu_o = 4\pi \times 10^{-7}$ permeability of free space (H/m)

$\epsilon = \epsilon_r \epsilon_o$ electric permittivity (F/m)

$\epsilon_o = 8.854 \times 10^{-12}$ permittivity of free space (F/m)

ϵ_r relative permittivity of material (dimensionless), ($\epsilon \approx 80$ for seawater)

σ electric conductivity (S/m)

$c = (\mu_o \epsilon_o)^{-1/2}$ speed of light $\approx 3 \times 10^8$ (m/s)

$K = (\sigma \mu_o)^{-1}$ magnetic diffusivity (m²/s)

$\Omega = 2\pi/1 \text{ day} \approx 7.3 \times 10^{-5}$ (radians/s) rotation rate of the earth

$R_E \approx 6370$ km, radius of earth

D_t total derivative (time rate of change moving with the fluid)

$\mathbf{u} = u\hat{x} + v\hat{y} + w\hat{z}$ velocity of ocean currents

ϕ latitude

Units

$$S = \text{siemens} = A/V$$

$$Wb = \text{webers} = V \cdot s$$

$$H = \text{henries} = Wb/A$$

$$F = \text{farads} = C/V$$

$$C = \text{coulombs}$$

$$A = \text{amperes} = C/s$$

$$T = \text{teslas} = Wb/m^2$$

$$V = \text{volts}$$

$$s = \text{seconds}$$

$$m = \text{meters}$$

2 Introduction

In this report series we investigate the theory of electromagnetic fields induced by ocean currents. The general motivation for this research is to explore the potential for using the existing magnetic data set in ocean and climate studies.

In the first report (Tyler and Mysak (1993)) we discussed the magnetohydrodynamics of a fluid with the conductivity and mass density characteristics of seawater. In this restricted case the electromagnetic body forces on this fluid due to its motion through a weak background magnetic field can be neglected and the conductivity and velocity fields can be prescribed.

We pointed out in Tyler and Mysak (1993) that the equations and results presented were not immediately applicable to the ocean since all calculations assumed an *inertial* reference frame rather than the realistic *accelerating* reference frame of the earth.

The purpose of this report is to generalize the earlier description of induction by ocean currents so that it includes the possible effects of rotation. It is expected that this additional formalism is required before any general consideration of the electrodynamic of large-scale ocean currents can be made.

By way of analogy, it is known that equations describing fluid dynamics in laboratories attached to the spinning earth can neglect the effects arising from rotation. Over the larger scales of the oceans, however, additional rotational terms must be included in the governing equations. Similarly, the success of certain sets of equations in describing the electrodynamic in laboratory experiments does not preclude the necessity for a formal consideration of rotational electrodynamic before application to the ocean scales can be justified.

2.1 Paradox of a Rotating Shell of Charge

Imagine placing a net charge onto a spherical shell and allowing the sphere to rotate. In an inertial reference frame, it is quite clear that the the rotating charge

can be treated as an electrical current and hence a magnetic field will be observed.

Now imagine making observations while moving in a frame that rotates with the shell. Now the electrical charges appear to be at rest so it could be wrongly assumed that no magnetic field is observed.

In fact, the magnetic field does not disappear for the rotating observer. Under typical experiments, we can even expect that the axisymmetric part (at least) of the magnetic field will be similar in both the rotating and stationary reference frames. A slightly more complicated version of this example is usually referred to as *Schiff's Paradox*. (For related discussion see Pegram (1917), Schiff (1939), Feynman (1964), Post and Bahulikar (1971), Corum (1980), and Lorrain (1993).)

At the heart of this paradox is the fact that the rotating reference frame is accelerating and equations of electrodynamics valid in an inertial frame do not carry over—even if the velocities involved in the rotation appear to be ‘non-relativistic’ ($|u| \ll c$).

Furthermore, it should be remembered that Special Relativity is not immediately extendable to accelerating reference frames and thus approximations arrived at by examining terms in the Lorentz transformation will not generally be valid in the accelerating (or rotating) reference frame. Stated another way, in the accelerating frame we can expect effects other than just the length contractions and time dilations predicted by Special Relativity.

Neither the constitutive relationships nor Maxwell’s equations are generally invariant under transformation to a rotating reference frame (see, for example, Feynman *et al* (1964), Van Bladel (1984), Schieber (1986)). Though this has been evident for much time, it has not always been appreciated, as noted in Webster and Whitten (1973): “However, the belief that all four of the field (*Maxwell's*) equations are invariant under such conditions ($|u| \ll c$) is still prevalent and causes misconceptions in physical applications, including astrophysical and geophysical ones.”

3 Electrodynamic Equations in a Rotating Coordinate System

Despite the fact that several papers have appeared regarding the proper form of Maxwell's equations in a rotating coordinate system, results are often confusing and difficult to compare for several reasons. First, results obtained for vacuum electrodynamics are not necessarily valid when material media is included. Similarly, certain assumptions about the medium such as homogeneity, small rotation velocities, or rotational symmetry, are made that limit the scope of validity of the results. Also, the transformation of Maxwell's equations will depend on their form (Amperian or Minkowskiian). Finally, there is the usual confusion stemming from different conventions on units.

Schiff (1939) used methods from General Relativity to transform the vacuum Maxwell's equations into the rotating form. Modesitt (1970) arrives at a similar restricted result using what he claims to be a "classical" derivation.

A clear and insightful derivation of Maxwell's equations in a relativistic rotating vacuum reference frame is given in Ise and Uretsky (1959, 1961).

When considering material media, extra terms appearing due to rotation have sometimes been described as constitutive in nature (Post and Bahulikar (1971)) and at other times they are described in terms of "fictitious" charges and currents (Schiff (1939), Webster (1963)).

Perhaps the greatest confusion stems from the transformation of the constitutive relationships to the rotating reference frame. (This can also play a role in the description of the rotating Maxwell's equations.) Anderson and Ryon (1969) claim that the work by Post (1967), Post and Yildiz (1965), and Yildiz and Tang (1966) (after here referred to as PYT) involves an *ad hoc* assumption since the

free-space and matter portions of the constitutive relationships have been transformed in different ways. Anderson and Ryon show that these earlier works, while satisfying one important condition (for a medium stationary in an inertial frame, the equations reduce to the familiar Maxwell-Minkowski form), they fail to satisfy the relativistic velocity addition law. Anderson and Ryon present a method which removes these inconsistencies.

In physical terms, the major criticism against the PYT scheme is that they implicitly assume that the state of motion of the medium has no physical consequences for electromagnetic radiation (Anderson and Ryon, (1969)).

Mo (1970) presents a detailed transformation of the equations of electrodynamics using a covariant formalism but his “free-space” treatment of one of the constitutive relationships was criticized by Post and Bahulikar (1971) as being contrary to experimental evidence.

In recent years, the discussion of rotational electrodynamics has become more clear (Van Bladel (1984), Schieber (1986), Kretzschmar and Fugmann (1989), Mashhoon (1989)) and consistency is being gained in the treatment of the constitutive relationships in accelerating reference frames using a “co-moving” (Van Bladel (1984)) or “locality” (Mashhoon (1988,1989)) hypothesis.

The general problem of rotational electrodynamics in the presence of material media is still not closed (P. Lorrain (1993) *pers. comm.*; B. Mashhoon (1994) *pers. comm.*). In short, there seems to be agreement on how to do electrodynamics for an inertial observer observing rotating material media; and there seems to be agreement on how to do electrodynamics in a rotating frame when no material is present. As of yet, there are still no undisputed methods of putting these two together. For our applications, however, we need to consider both formulations of electrodynamics valid for observers in the rotating frame, and the presence of a rotating non-homogenous medium.

Fortunately, however, the remaining disagreement between different researcher’s results usually appear in the second-order $(u/c)^2$ terms. So for the case $(u/c)^2 \ll$

1 that we will be considering, a greater consistency among results from different researchers does occur.

We shall follow the style of the later papers (particularly those by Van Bladel) in presenting our formulation of the equations describing electrodynamics in a rotating coordinate system attached to the earth's surface.

Briefly the strategy is as follows. Upon requiring covariance of Maxwell's equations in their tensorial form, and using the metric describing the rotating coordinate system, we derive the vector form of Maxwell's equations in the rotating frame. We then use the co-moving hypothesis (to be described below) to determine the transformations of the constitutive relationships.

3.1 Maxwell's Equations

In the following, we will use upper-case letters to denote quantities observed in an inertial reference frame and lower-case letters for quantities observed in the rotating reference frame.

In general relativity theory, the principle of equivalence states that noninertial reference frames are equivalent to a gravitational field. This should be kept in mind when reviewing the literature since much of what we will be presenting was developed in the gravitational literature and has been adopted for rotating (noninertial) systems. In fact, while discussing the effect of a gravitational field on Maxwell's equations, Van Bladel also derives the transformation to rotating cylindrical coordinates.

Briefly, the electromagnetic tensors $M_{\alpha\beta}$, $N_{\alpha\beta}$ satisfying the generalized four-dimensional Maxwell's equations

$$\frac{1}{|g|} \sum_{\beta=0}^3 \frac{\partial}{\partial x^\beta} (\sqrt{|g|} M^{\alpha\beta}) = J^\alpha, \quad (1)$$

$$\frac{\partial N_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial N_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial N_{\gamma\alpha}}{\partial x^\beta} = 0, \quad (2)$$

under an arbitrary metric $g_{\alpha\beta}$ are given by

$$M^{\alpha\beta} = \begin{bmatrix} 0 & c \frac{d^1}{\sqrt{g_{00}}} & c \frac{d^2}{\sqrt{g_{00}}} & c \frac{d^3}{\sqrt{g_{00}}} \\ -c \frac{d^1}{\sqrt{g_{00}}} & 0 & \frac{h_3}{\sqrt{|g|}} & -\frac{h_2}{\sqrt{|g|}} \\ -c \frac{d^2}{\sqrt{g_{00}}} & -\frac{h_3}{\sqrt{|g|}} & 0 & \frac{h_1}{\sqrt{|g|}} \\ -c \frac{d^3}{\sqrt{g_{00}}} & \frac{h_2}{\sqrt{|g|}} & -\frac{h_1}{\sqrt{|g|}} & 0 \end{bmatrix}, \quad (3)$$

$$N_{\alpha\beta} = \begin{bmatrix} 0 & -e_1/c & -e_2/c & -e_3/c \\ e_1/c & 0 & \sqrt{|\gamma|}b^3 & -\sqrt{|\gamma|}b^2 \\ e_2/c & -\sqrt{|\gamma|}b^3 & 0 & \sqrt{|\gamma|}b^1 \\ e_3/c & \sqrt{|\gamma|}b^2 & -\sqrt{|\gamma|}b^1 & 0 \end{bmatrix}. \quad (4)$$

Above, $J^\alpha = (\rho_e c, \mathbf{j})$ is the four current, $-|g|$ is the determinant of $g_{\alpha\beta}$ and $|\gamma|$ is the determinant of γ_{ik} , where $\gamma_{ik} = g_i g_k - g_{ik}$ and $g_i = g_{0i}(g_{00})^{-1/2}$. Here, a useful relationships is $|g| = g_{00}|\gamma|$. Also, while greek indices refer to summation over 0,1,2,3, latin indices sum over just 1,2,3. Letters with superscripts refer to contravariant components and subscripts refer to covariant components of the vectors \mathbf{e} , \mathbf{b} , and \mathbf{h} (described in the list of symbols).

The metric describing a rotating cylindrical coordinate system is

$$g_{\alpha\beta} = \begin{bmatrix} 1 - \frac{\Omega^2 r^2}{c^2} & 0 & \frac{-\Omega r^2}{c} & 0 \\ 0 & -1 & 0 & 0 \\ \frac{-\Omega r^2}{c} & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (5)$$

in the above, we have neglected the gravitational effects of the electromagnetic field on the metric tensor $g_{\alpha\beta}$.

Using 5 in 3 and 4, we obtain the following electromagnetic tensors in rotating cylindrical coordinates:

$$M^{\alpha\beta} = \begin{bmatrix} 0 & cd^r(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} & cd^\theta(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} & cd^z(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} \\ -cd^r(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} & 0 & h_z/r & -h_\theta/r \\ -cd^\theta(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} & -h_z/r & 0 & h_r/r \\ -cd^z(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} & h_\theta/r & -h_r/r & 0 \end{bmatrix}, \quad (6)$$

$$N_{\alpha\beta} = \begin{bmatrix} 0 & -e_r/c & -e_\theta/c & -e_z/c \\ e_r/c & 0 & rb^z(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} & -rb^\theta(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} \\ e_\theta/c & -rb^z(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} & 0 & rb^r/(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} \\ e_z/c & rb^\theta(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} & -rb^r(1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}} & 0 \end{bmatrix}. \quad (7)$$

Inserting 6 and 7 in 1 and 2, neglecting terms in $(\frac{u}{c})^2 = (\frac{\Omega r}{c})^2$ and converting to physical-component vector form, Van Bladel recovers the usual Maxwell's equations in cylindrical polar coordinates:

$$\nabla \times \mathbf{e} = -\partial_t \mathbf{b}, \quad (8)$$

$$\nabla \cdot \mathbf{d} = \rho_e, \quad (9)$$

$$\nabla \times \mathbf{h} = \partial_t \mathbf{d} + \mathbf{j} \quad (10)$$

$$\nabla \cdot \mathbf{b} = 0. \quad (11)$$

We see that *this form* of Maxwell's equations appears to retain its form under transformation to a slowly rotating coordinate system. This will not necessarily be true for other forms. Also, although Maxwell's equations have a similar form, this does not mean that measurements of the fields and sources would be the same as those measured by an inertial observer.

3.2 Constitutive Relationships in a Rotating Coordinate System

As discussed above, the usual constitutive relationships of the form $\mathbf{D} = \epsilon \mathbf{E}$, for example, do not hold in the rotating reference frame. Instead, for this example, we must consider a relationship of the form $\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B})$ or $\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H})$.

In fact, consideration of the motion of the media is necessary even when the motions are uniform translations. Furthermore, as pointed out in Bolotovskii and Stolyarov (1975), the ratio of the velocity of the medium to c is not always the parameter that determines the importance of relativistic effects when material media is present. They cite examples of so-called "moderating systems",

in which the speed of propagation of electromagnetic waves can be significantly smaller than the speed of light in vacuum leading in some cases to relativistic effects even when the velocity of the medium is quite small.

In this section we will consider two cases. In both cases the observer is moving in slow solid-body rotation. In the first case that we consider, the material medium is rotating with the observer. Hence, this case gives the constitutive relationships for material that appears stationary to the rotating observer.

In the second case, the material has a velocity relative to the observer. This is relevant since we wish to derive the appropriate constitutive relationships to be used by an observer moving with the solid earth who observes the ocean moving with a different velocity.

3.2.1 Co-Moving Hypothesis

A frame of reference that is “co-moving” with another reference frame has, during that instant, no relative velocity with the latter. The hypothesis of “locality” or the “co-moving” assumption (also called the “instantaneous rest-frame” theory (Van Bladel (1976))) proposes that local observations made by an accelerated observer are identical to those made by an observer that is instantly at rest in the so-called “co-moving” frame. This idea can be made more clear with an example.

Consider a spherical-shell coordinate system attached to the surface of a spinning globe. This frame must normally be accelerating if it is to remain attached to the surface of the globe. Now imagine that at time $t = t_1$ we define two different coordinate frames: one which continues attached to the surface for $t > t_1$; and a second which is allowed to ‘fly apart’. By ‘fly apart’ we mean that for $t > t_1$ the second system will no longer be accelerating. Each element of the coordinate system will be allowed to follow a steady motion with the velocity it had at $t = t_1$.

Now at $t = t_1 + \delta t$ ($\delta t \rightarrow 0$) we describe the second reference frame as a non-accelerating frame that is instantaneously co-moving with the first accelerated reference frame.

The hypothesis of locality claims that for phenomena involving no extensions in space-time the results measured in both reference frames described above should be identical. (By “no extensions in space time” we mean that the measurements must be point-like in space-time (coincidences); comparisons involving two different points in space or at different times are not in this category.) The accelerated reference frame, in this respect, is seen as a collection of inertial reference frames which vary in space and time.

The co-moving hypothesis should be understood as an approximation. In particular, Van Bladel (1976) draws attention to the following shortcomings:

- It neglects the coupling between gravitational fields and electromagnetic fields.
- It neglects the influence of the centrifugal forces, which tend to deform the material and to create anisotropies and nonlinearities.
- It does not recognize the fact that in general an accelerated medium will not be in local equilibrium, and hence the conditions for the derivation of linear laws are not respected.

The above considerations will not be important in our application. This can be easily seen by criteria developed by Shiozawa (1973) (but see letter by Atwater (1975) regarding this study and following rebuttal by Shiozawa).

3.2.2 Approximate Constitutive Relationships in a Rotating Coordinate System (Material Medium in Solid-Body Rotation)

Under the co-moving hypothesis, Van Bladel derives the constitutive relationships for material media under rotation. Neglecting any anisotropy in the electrical

properties of the material that might arise from stresses and deformation of the material due to rotation, he writes for the co-moving reference frame,

$$\mathbf{D}' = \epsilon \mathbf{E}' \quad (12)$$

$$\mathbf{B}' = \mu \mathbf{H}', \quad (13)$$

$$\mathbf{J}' = \sigma \mathbf{E}'. \quad (14)$$

Above, $\epsilon = \epsilon_r \epsilon_o$ is the absolute electrical permittivity, ϵ_r is the relative permittivity, ϵ_o is the permittivity of free space, $\mu = \mu_r \mu_o$ is the magnetic permeability, μ_r is the relative permeability, μ_o is the permeability of free space, and σ is the conductivity.

To find the components of the constitutive relationships in the accelerating frame we could proceed by first relating the metric of the two systems (Heer (1964)) and then seeking a direct transformation. For clarity, however, the two-step approach used by Van Bladel is preferred. In the first step, a transformation is made from the co-moving inertial frame to a stationary inertial frame. That is, the co-moving frame is defined by the instantaneous velocity field of the accelerating frame. The transformation takes us to the frame in which these velocities vanish. This transformation is then similar to the simple Lorentz transformation but with the velocities being a function of space. Hence, we are transforming a collection of inertial frames to a common inertial frame. We then have the constitutive relationships of the rotating media as they would be observed by an observer in a rigid inertial reference frame.

In the second step, we transform the components of the constitutive relationships from the inertial frame to the accelerating (rotating) frame in a manner similar to that shown for Maxwell's equations.

This calculation is shown in Van Bladel (1984). After assuming $(\frac{u}{c})^2 = (\frac{\Omega r}{c})^2 \ll 1$ and writing his results in a vectorial form comparable with that of Anderson and Ryon (1969), we have (for quantities in the rotating system)

$$\mathbf{d} = \epsilon[\mathbf{e} - \frac{1}{N^2} \mathbf{u} \times \mathbf{b}], \quad (15)$$

$$\mathbf{h} = \frac{1}{\mu} \mathbf{b} - \frac{\epsilon}{N^2} \mathbf{u} \times \mathbf{e}, \quad (16)$$

where $N = (\mu_r \epsilon_r)^{1/2}$ is the index of refraction.

These results agree with those calculated by Anderson and Ryon (1969). For nonmagnetic material $\mu = \mu_o$ and since we are neglecting the second-order terms, the results of PYT are also consistent.

In a manner similar to that above, Ohm's law is seen to carry over to the rotating system:

$$\mathbf{j} = \sigma \mathbf{e}. \quad (17)$$

Now we can use equations 15-17 to remove \mathbf{D} and \mathbf{H} from the Minkowskiian Maxwell's equation 8-11. This gives

$$\nabla \times \mathbf{e} = -\partial_t \mathbf{b}, \quad (18)$$

$$\nabla \cdot \mathbf{b} = 0, \quad (19)$$

$$\nabla \cdot (\epsilon \mathbf{e} - \frac{\epsilon}{N^2} \mathbf{u} \times \mathbf{b}) = \rho_e, \quad (20)$$

$$\nabla \times (\frac{1}{\mu} \mathbf{b} - \frac{\epsilon}{N^2} \mathbf{u} \times \mathbf{e}) = \partial_t (\epsilon \mathbf{e} - \frac{\epsilon}{N^2} \mathbf{u} \times \mathbf{b}) + \mathbf{j}. \quad (21)$$

3.2.3 Approximate Constitutive Relationships in a Rotating Coordinate System (Material Medium with Generalized Velocity)

The constitutive relationships derived above will be inadequate to describe processes in the ocean since in this case the medium (ocean) has a velocity relative to the rotating observer.

Considering the arguments used in the co-moving hypothesis, the constitutive equations derived for the solid-body rotation medium could as easily have been given for an observer being advected along by an ocean current. The velocities in this case would represent the velocity of the solid-body rotation plus the velocity of the ocean currents.

This suggests a means of deriving the appropriate constitutive relationships for the ocean. Then we have two steps. First, we assume relationships of the form 12—14 in a frame instantaneously at rest with the oceans. As before we Lorentz transform this collection of inertial frames to a common inertial frame stationary with respect to the rotating earth.

The second step is the same as before; we transform all quantities to the accelerating (rotating) reference frame as we did for the rotational Maxwell's equations and constitutive relationships for the material media in solid-body rotation.

After doing this, we find we can write the constitutive equations in a form similar to 15—16:

$$\mathbf{d} = \epsilon[\mathbf{e} - \frac{1}{N^2}\tilde{\mathbf{u}} \times \mathbf{b}], \quad (22)$$

$$\mathbf{h} = \frac{1}{\mu}\mathbf{b} - \frac{\epsilon}{N^2}\tilde{\mathbf{u}} \times \mathbf{e}, \quad (23)$$

where

$$\tilde{\mathbf{u}} = \mathbf{u}_s + \mathbf{u}_c(1 - N^2), \quad (24)$$

in which \mathbf{u}_c is the ocean current velocity with respect to the solid earth, and the solid-body rotation velocity of the earth is

$$\mathbf{u}_s = \Omega r \cos \phi \hat{\lambda} = \Omega r \sin \theta \hat{\lambda}, \quad (25)$$

where ϕ is the latitude and θ is the colatitude.

The relation between \mathbf{j} and \mathbf{e} becomes

$$\mathbf{j} = \rho_e \mathbf{u}_c + \sigma(\mathbf{e} + \mathbf{u}_c \times \mathbf{b}). \quad (26)$$

Equation 26 states that an observer attached to the solid earth observes electrical currents due to space charges ρ_e advected by the moving medium (in this case the moving medium is the ocean and has relative velocity \mathbf{u}_c) as well as the usual currents induced by the electrical field observed in the frame of reference moving with the medium.

In deriving 22—26 we neglected $\frac{(|\mathbf{u}_s|+|\mathbf{u}_c|)^2}{c^2}$, $\frac{|\mathbf{u}_s||\mathbf{u}_c|}{c^2}$, $\frac{|\mathbf{u}_c|^2}{c^2}$ and $N^{-2} \frac{(|\mathbf{u}_s|+|\mathbf{u}_c|)^2}{c^2}$ relative to 1.

Now we can use equations 22-26 to remove \mathbf{D} and \mathbf{H} from the Minkowskiian Maxwell's equation 8-11. This gives

$$\nabla \times \mathbf{e} = -\partial_t \mathbf{b}, \quad (27)$$

$$\nabla \cdot \mathbf{b} = 0, \quad (28)$$

$$\nabla \cdot (\epsilon \mathbf{e} - \frac{\epsilon}{N^2} \tilde{\mathbf{u}} \times \mathbf{b}) = \rho_e, \quad (29)$$

$$\nabla \times (\frac{1}{\mu} \mathbf{b} - \frac{\epsilon}{N^2} \tilde{\mathbf{u}} \times \mathbf{e}) = \partial_t (\epsilon \mathbf{e} - \frac{\epsilon}{N^2} \tilde{\mathbf{u}} \times \mathbf{b}) + \mathbf{j}. \quad (30)$$

3.3 Induction Equation in a Rotating Coordinate System

Equations 27-30 give Maxwell's equations in terms of E, B . It was assumed that the velocities involved were much smaller than that of light. No assumptions about the length or time scales have yet been made. We now wish to combine for a higher order governing equation involving only \mathbf{B} . To do this we will now make additional assumptions concerning the length and time scales. From now on, we will not be considering magnetized media, so we let $\mu = \mu_o$.

We start by considering equation 26. Using 29 to substitute for ρ_e , and setting $N^2 = \mu_r \epsilon_r = (1) \epsilon_r$, we can rewrite 26 as

$$\mathbf{j} = \mathbf{u}_c \epsilon_o [\nabla \cdot (\epsilon_r \mathbf{e} - \tilde{\mathbf{u}} \times \mathbf{b})] + \sigma (\mathbf{e} + \mathbf{u}_c \times \mathbf{b}). \quad (31)$$

Now we can see by comparing the terms in the square brackets with the terms in the curved brackets that the charge advection (square brackets) can be neglected compared to the charge conduction (curved brackets) provided the length scale we consider satisfies both $\mathcal{L} \gg \frac{\epsilon}{\sigma} |\mathbf{u}_c|$ and $\mathcal{L} \gg \frac{\epsilon \omega}{\sigma} |\tilde{\mathbf{u}}|$.

In another sense, it can be easily shown that in order for the magnetic fields due to charge advected by the rotating earth to be important, the space charge would be so large that the associated electric fields would be unrealistically large

(Stevenson (1974); Lorrain (1988)). With the charge advection term neglected, 31 becomes

$$\mathbf{j} = \sigma(\mathbf{e} + \mathbf{u}_c \times \mathbf{b}). \quad (32)$$

We can use 32 to write 30 as

$$\nabla \times \left(\frac{1}{\mu_o} \mathbf{b} \right) - \nabla \times (\epsilon_o \tilde{\mathbf{u}} \times \mathbf{e}) - \partial_t(\epsilon \mathbf{e}) + \partial_t(\epsilon_o \tilde{\mathbf{u}} \times \mathbf{b}) - \sigma \mathbf{e} - \sigma \mathbf{u}_c \times \mathbf{b} = 0, \quad (33)$$

where we have set $\mu_r = 1$.

If we consider times scales $\tau \gg \frac{\epsilon}{\sigma}$, we can neglect the third term relative to the fifth. Also, if $\tilde{\mathbf{u}}(\mathcal{L}/\tau)c^{-2} \ll 1$, the fourth term will be much smaller than the first. Finally, for $\mathcal{L} \gg \frac{\epsilon_o}{\sigma} \tilde{\mathbf{u}}$ the second term is small relative to the fifth. Considering the typical values $\epsilon = \epsilon_r \epsilon_o \approx 80 \epsilon_o$, $\epsilon_o = 8.854 \times 10^{-12}$ F/m, $c^2 = (\epsilon_o \mu_o)^{-1/2} \approx 9 \times 10^{16}$ (m/s), $\tilde{\mathbf{u}} \approx 10^2$ m/s, we can see that the conditions above are satisfied when studying induction by the ocean currents. The dominant balance is then between the curl of the magnetic field and the electrical current, as in the inertial frame case:

$$\nabla \times \mathbf{b} = \mu_o \mathbf{j} = \sigma(\mathbf{e} + \mathbf{u}_c \times \mathbf{b}). \quad (34)$$

Since 27, 28, 34 and 32 have forms similar to those found in the inertial frame, we can combine these into an induction equation in a manner similar to that done for the inertial frame (this can be seen in Tyler and Mysak (1993)). The induction equation is given in the appendix in vectorial form as well as for several common coordinate systems.

4 Summary

In summary, we have derived the equations of electrodynamics that should be used in a rotating reference frame. Both cases of a material medium stationary with respect to the rotating reference frame, and a medium moving in the rotating frame (such as the ocean does) can be described by the same set of equations once we define the velocity $\tilde{\mathbf{u}} = \mathbf{u}_s + (1 - N^2)\mathbf{u}_c$.

We can then write the low-velocity constitutive relationships for the rotating frame as

$$\mathbf{d} = \epsilon[\mathbf{e} - \frac{1}{N^2}\tilde{\mathbf{u}} \times \mathbf{b}], \quad (35)$$

$$\mathbf{h} = \frac{1}{\mu}\mathbf{b} - \frac{\epsilon}{N^2}\tilde{\mathbf{u}} \times \mathbf{e}, \quad (36)$$

$$\mathbf{j} = \rho_e \mathbf{u}_c + \sigma(\mathbf{e} + \mathbf{u}_c \times \mathbf{b}). \quad (37)$$

Maxwell's equations for \mathbf{e} and \mathbf{b} can then be written as

$$\nabla \times \mathbf{e} = -\partial_t \mathbf{b}, \quad (38)$$

$$\nabla \cdot \mathbf{b} = 0, \quad (39)$$

$$\nabla \cdot (\epsilon \mathbf{e} - \frac{\epsilon}{N^2}\tilde{\mathbf{u}} \times \mathbf{b}) = \rho_e, \quad (40)$$

$$\nabla \times (\frac{1}{\mu}\mathbf{b} - \frac{\epsilon}{N^2}\tilde{\mathbf{u}} \times \mathbf{e}) = \partial_t (\epsilon \mathbf{e} - \frac{\epsilon}{N^2}\tilde{\mathbf{u}} \times \mathbf{b}) + \mathbf{j}. \quad (41)$$

When the electrical properties of the ocean as well as the length scales are considered, equations 38, 39, and 41 are approximately equivalent to the usual inertial-frame form and an induction equation for \mathbf{b} can be derived and is given in the appendix.

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6 Appendix

In the following we give the induction equation in several vector forms as well as the component forms for common coordinate systems. We have shown that for the observer rotating with the earth and observing electrodynamics due to the oceans

moving with relative velocity, the appropriate induction equation has a form approximately similar to that for the inertial-observer case. For our applications \mathbf{u} should be understood to be the velocity of the ocean currents (earlier referred to as \mathbf{u}_c) with respect to the solid earth. The magnetic diffusion coefficient is $K = (\mu_o\sigma)^{-1}$. For a description of the physical significance of the individual terms see Tyler and Mysak (1993).

6.1 Vector Induction Equation

It is possible to write the induction equation in several ways. While the different forms are of course equivalent, some forms are more amenable for certain applications. More importantly, the finite-difference forms of these equations are not equivalent and we have derived a ‘flux’ form (equation 47) for use in the discussion of numerical studies to be presented later.

As described in Tyler and Mysak (1993), we are primarily interested in steady-state solutions. The standard form of the induction equation we will use is shown in 47. This form is useful since for steady state numerical problems, the equation can be divided through by K , and we avoid the ∇K terms which can be very large at conductivity interfaces.

In general, we can write the induction equation in the following forms.

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B} - K \nabla \times \mathbf{B}), \quad (42)$$

$$\partial_t \mathbf{B} = \{ -(\nabla \cdot \mathbf{u}) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{B} \} - \nabla \times (K \nabla \times \mathbf{B}). \quad (43)$$

$$D_t \mathbf{B} = \partial_t \mathbf{B} - (\mathbf{u} \cdot \nabla) \mathbf{B} = \{ -(\nabla \cdot \mathbf{u}) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{u} \} - \nabla \times (K \nabla \times \mathbf{B}). \quad (44)$$

When we restrict ourselves to Cartesian coordinates, we can use the identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ together with the nondivergence of \mathbf{B} to write

$$D_t \mathbf{B} = \partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} = -(\nabla \cdot \mathbf{u}) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{u} + K \nabla^2 \mathbf{B} + K \nabla \ln \sigma \times (\nabla \times \mathbf{B}), \quad (45)$$

or in a flux form as

$$\begin{aligned}
\partial_t \mathbf{B} &= \nabla \cdot \{-u\mathbf{B} + B_x \mathbf{u} + K\nabla B_x - K\partial_x \mathbf{B}\} \hat{x} \\
&+ \nabla \cdot \{-v\mathbf{B} + B_y \mathbf{u} + K\nabla B_y - K\partial_y \mathbf{B}\} \hat{y} \\
&+ \nabla \cdot \{-w\mathbf{B} + B_z \mathbf{u} + K\nabla B_z - K\partial_z \mathbf{B}\} \hat{z}.
\end{aligned} \tag{46}$$

We can usually assume the ocean to be incompressible, hence 45 becomes

$$D_t \mathbf{B} = \partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u} + K\nabla^2 \mathbf{B} + K\nabla \ln \sigma \times (\nabla \times \mathbf{B}). \tag{47}$$

Writing 47 out for the three components, we have

$$\begin{aligned}
\partial_t B_x + u\partial_x B_x + v\partial_y B_x + w\partial_z B_x &= B_x \partial_x u + B_y \partial_y u + B_z \partial_z u \\
&+ K\partial_x \partial_x B_x + K\partial_y \partial_y B_x + K\partial_z \partial_z B_x \\
&+ K\partial_y \ln \sigma (\partial_x B_y - \partial_y B_x) - K\partial_z \ln \sigma (\partial_z B_x - \partial_x B_z)
\end{aligned} \tag{48}$$

$$\begin{aligned}
\partial_t B_y + u\partial_x B_y + v\partial_y B_y + w\partial_z B_y &= B_x \partial_x v + B_y \partial_y v + B_z \partial_z v \\
&+ K\partial_x \partial_x B_y + K\partial_y \partial_y B_y + K\partial_z \partial_z B_y \\
&+ K\partial_x \ln \sigma (\partial_y B_x - \partial_x B_y) - K\partial_z \ln \sigma (\partial_z B_y - \partial_y B_z)
\end{aligned} \tag{49}$$

$$\begin{aligned}
\partial_t B_z + u\partial_x B_z + v\partial_y B_z + w\partial_z B_z &= B_x \partial_x w + B_y \partial_y w + B_z \partial_z w \\
&+ K\partial_x \partial_x B_z + K\partial_y \partial_y B_z + K\partial_z \partial_z B_z \\
&+ K\partial_x \ln \sigma (\partial_z B_x - \partial_x B_z) - K\partial_y \ln \sigma (\partial_y B_z - \partial_z B_y)
\end{aligned} \tag{50}$$

In the following, the induction equation assuming incompressible flow will be given in spherical and cylindrical coordinates. Expressions for the curl and scalar Laplacian terms for either coordinate system can be found in standard texts. The subscript on the curl terms describe the component (not partial differentiation).

6.2 Induction Equation in Spherical Coordinates (for colatitude θ)

r equation:

$$\begin{aligned} \partial_t B_r + [\mathbf{u} \cdot \nabla B_r] = & [\mathbf{B} \cdot \nabla u_r] \\ & + K \left[\nabla^2 B_r - \frac{2}{r^2} B_r - \frac{2}{r^2 \sin \theta} \partial_\theta (B_\theta \sin \theta) - \frac{2}{r^2 \sin \theta} \partial_\lambda B_\lambda \right] \\ & + K \left[\frac{1}{r} \partial_\theta \ln \sigma (\nabla \times \mathbf{B})_\lambda - \frac{1}{r \sin \theta} \partial_\lambda \ln \sigma (\nabla \times \mathbf{B})_\theta \right] \end{aligned} \quad (51)$$

θ equation:

$$\begin{aligned} \partial_t B_\theta + \left[\mathbf{u} \cdot \nabla B_\theta + \frac{1}{r} u_\theta B_r \right] = & \\ & \left[\mathbf{B} \cdot \nabla u_\theta + \frac{1}{r} B_\theta u_r \right] \\ & + K \left[\nabla^2 B_\theta + \frac{2}{r^2} \partial_\theta B_r - \frac{1}{(r \sin \theta)^2} B_\theta - 2 \frac{1}{r^2} \frac{\cos \theta}{\sin^2 \theta} \partial_\lambda B_\lambda \right] \\ & + K \left[\frac{1}{r \sin \theta} \partial_\lambda \ln \sigma (\nabla \times \mathbf{B})_r - \partial_r \ln \sigma (\nabla \times \mathbf{B})_\lambda \right] \end{aligned} \quad (52)$$

λ equation:

$$\begin{aligned} \partial_t B_\lambda + \left[\mathbf{u} \cdot \nabla B_\lambda + \frac{1}{r} u_\lambda B_r + \frac{1}{r} u_\lambda B_\theta \cot \theta \right] = & \\ & \left[\mathbf{B} \cdot \nabla u_\lambda + \frac{1}{r} B_\lambda u_r + \frac{1}{r} B_\lambda u_\theta \cot \theta \right] \\ & + K \left[\nabla^2 B_\lambda + 2 \frac{1}{r^2 \sin \theta} \partial_\lambda B_r + 2 \frac{\cos \theta}{(r \sin \theta)^2} \partial_\lambda B_\theta - \frac{1}{(r \sin \theta)^2} B_\lambda \right] \\ & + K \left[\partial_r \ln \sigma (\nabla \times \mathbf{B})_\theta - \frac{1}{r} \partial_\theta \ln \sigma (\nabla \times \mathbf{B})_r \right] \end{aligned} \quad (53)$$

6.3 Induction Equation in Spherical Coordinates (for latitude ϕ)

r equation:

$$\begin{aligned} \partial_t B_r + [\mathbf{u} \cdot \nabla B_r] = & [\mathbf{B} \cdot \nabla u_r] \\ & + K \left[\nabla^2 B_r - \frac{2}{r^2} B_r - \frac{2}{r^2 \cos \phi} \partial_\phi (B_\phi \cos \phi) - \frac{2}{r^2 \cos \phi} \partial_\lambda B_\lambda \right] \\ & + K \left[-\frac{1}{r} \partial_\phi \ln \sigma (\nabla \times \mathbf{B})_\lambda + \frac{1}{r \cos \phi} \partial_\lambda \ln \sigma (\nabla \times \mathbf{B})_\phi \right] \end{aligned} \quad (54)$$

ϕ equation:

$$\begin{aligned} \partial_t B_\phi + \left[-\mathbf{u} \cdot \nabla B_\phi - \frac{1}{r} u_\phi B_r \right] = & \\ \left[-\mathbf{B} \cdot \nabla u_\phi - \frac{1}{r} B_\phi u_r \right] & \\ + K \left[-\nabla^2 B_\phi - \frac{2}{r^2} \partial_\phi B_r + \frac{1}{(r \cos \phi)^2} B_\phi - 2 \frac{1}{r^2} \frac{\sin \phi}{\cos^2 \phi} \partial_\lambda B_\lambda \right] & \\ + K \left[\frac{1}{r \cos \phi} \partial_\lambda \ln \sigma (\nabla \times \mathbf{B})_r - \partial_r \ln \sigma (\nabla \times \mathbf{B})_\lambda \right] & \end{aligned} \quad (55)$$

λ equation:

$$\begin{aligned} \partial_t B_\lambda + \left[\mathbf{u} \cdot \nabla B_\lambda + \frac{1}{r} u_\lambda B_r - \frac{1}{r} u_\lambda B_\phi \tan \phi \right] = & \\ \left[\mathbf{B} \cdot \nabla u_\lambda + \frac{1}{r} B_\lambda u_r - \frac{1}{r} B_\lambda u_\phi \tan \phi \right] & \\ + K \left[\nabla^2 B_\lambda + 2 \frac{1}{r^2 \cos \phi} \partial_\lambda B_r - 2 \frac{\sin \phi}{(r \cos \phi)^2} \partial_\lambda B_\phi - \frac{1}{(r \cos \phi)^2} B_\lambda \right] & \\ + K \left[-\partial_r \ln \sigma (\nabla \times \mathbf{B})_\phi + \frac{1}{r} \partial_\phi \ln \sigma (\nabla \times \mathbf{B})_r \right] & \end{aligned} \quad (56)$$

6.4 Induction Equation in Cylindrical Coordinates

z equation:

$$\begin{aligned} \partial_t B_z + [\mathbf{u} \cdot \nabla B_z] = & [(\mathbf{B} \cdot \nabla) u_z] + K [\nabla^2 B_z] \\ & + K \left[\partial_r \ln \sigma (\nabla \times \mathbf{B})_\lambda - \frac{1}{r} \partial_\lambda \ln \sigma (\nabla \times \mathbf{B})_r \right] \end{aligned} \quad (57)$$

r equation:

$$\begin{aligned} \partial_t B_r + [\mathbf{u} \cdot \nabla B_r] = & [\mathbf{B} \cdot \nabla u_r] \\ & + K \left[\nabla^2 B_r - \frac{1}{r^2} B_r - 2 \frac{1}{r^2} \partial_\lambda B_\lambda \right] + K \left[\frac{1}{r} \partial_\lambda \ln \sigma (\nabla \times \mathbf{B})_z - \partial_z \ln \sigma (\nabla \times \mathbf{B})_\lambda \right] \end{aligned} \quad (58)$$

λ equation:

$$\begin{aligned} \partial_t B_\lambda + \left[\mathbf{u} \cdot \nabla B_\lambda + \frac{1}{r} u_\lambda B_r \right] = & \left[\mathbf{B} \cdot \nabla u_\lambda + \frac{1}{r} B_\lambda u_r \right] \\ & + K \left[\nabla^2 B_\lambda + 2 \frac{1}{r^2} \partial_\lambda B_r - \frac{1}{r^2} B_\lambda \right] + K [\partial_z \ln \sigma (\nabla \times \mathbf{B})_r - \partial_r \ln \sigma (\nabla \times \mathbf{B})_z] \end{aligned} \quad (59)$$

7 References

Anderson J.L. and J.W. Ryon (1969) Electromagnetic Radiation in Accelerated Systems. *Physical Review*, 181, 1765-1775.

Atwater, H.A. (1975) The Electromagnetic Field in Rotating Coordinate Frames. *Proceedings of the IEEE*, 63, 316-318.

Bolotovskii B.M. and S.N. Stolyarov (1975) Current Status of the Electrodynamics of Moving Media (Infinite Media). *Soviet Physics—Usp.*, 17, 875—895.

Corum, James F. (1980) Relativistic Covariance and Rotational Electrodynamics. *Journal of Mathematical Physics*, 21, 2360-2362.

Feynman, R.P., R.B. Leighton and M. Sands. (1964) *The Feynman Lectures on Physics*, Vol. II, Addison-Wesley, Reading, Mass.

Heer, C.V. (1964) Resonant Frequencies of an Electromagnetic Cavity in an Accelerated System of Reference. *Physical Review*, 134, A799-805.

Ise, John, and Jack L. Uretsky (1959) Vacuum Electrodynamics on a Merry-Go-Round. *American Journal of Physics*, 26, 431-435.

Ise, John, and Jack L. Uretsky (1961) Erratum: Vacuum Electrodynamics on a Merry-Go-Round. *American Journal of Physics*, 29, 329.

Kretzschmar, M. and W. Fugman (1989) The Electromagnetic Field of an Accelerated Charge in the Proper Reference Frame of a Noninertial Observer. *II Nuovo Cimento*, 103B, 389-412.

Lorrain, Paul (1993) Azimuthal Magnetic Fields in the Earth's Core. *Physica Scripta*, 47, 461—468.

Lorrain, P., D.R. Corson and F. Lorrain (1988) *Electromagnetic Field and Waves*, W.H. Freeman, New York, 3rd edn.

Mashhoon, Bahram (1988) Neutron Interferometry in a Rotating Frame of Reference. *Physical Review Letters* 61, 2639—2642.

Mashhoon, Bahram (1989) Electrodynamics in a Rotating Frame of Reference. *Physics Letters A*, 139, 103—139.

Mo, Tse Chin (1970) Theory of Electrodynamics in Media in Noninertial Frames and Applications. *Journal of Mathematical Physics*, 11 2589-2610.

Modesitt, G.E. (1970) Maxwell's Equations in a Rotating Reference Frame. *American Journal of Physics*, 38, 1487-1489.

- Pegram, G.B. (1917) Unipolar Induction and Electron Theory, *Physical Review*, 10, 591-600.
- Post, E.J. and D.D. Bahulikar (1971) Note on the Electrodynamics of Accelerated Systems. *Journal of Mathematical Physics*, 12, 1098-1099.
- Post, E.J. and Asim Yildiz (1965) Cavity Resonances in Accelerated Systems. *Physical Review Letters*, 15, 177—178.
- Schieber, D. (1986) *Electromagnetic Induction Phenomena*. Springer Series in Electrophysics, vol. 16. Springer-Verlag, New York, 296 pp.
- Schiff L.I. (1939) A Question in General Relativity. *Proceedings of the National Academy of Science USA*. 25, 391-395.
- Shiozawa, T. (1973) Phenomenological and Electron-Theoretical Study of the Electrodynamics of Rotating Systems. *Proceedings of the IEEE*, 61, 1694-1702.
- Stevenson, David (1974) Planetary Magnetism. *Icarus*, 22, 403—415.
- Tyler, Robert H. and Lawrence A. Mysak (1993) The Potential for Using Geomagnetic Data in Ocean and Climate Studies. I: Theory of Electromagnetic Fields Induced by Ocean Currents. *Centre for Climate and Global Change Research*, Report no. 93-20, McGill University, Montreal, Quebec, Canada.
- Van Bladel, J. (1976) Electromagnetic Fields in the Presence of Rotating Bodies, *Proceedings of the IEEE*, 64, 301—318.

Van Bladel, J. (1984) *Relativity and Engineering*, Springer Series in Electrodynamics, vol. 15. Springer-Verlag, New York, 402 pp.

Webster, David L. (1963) Schiff's Charges and Currents in Rotating Matter. *American Journal of Physics*, 31, 590-597.

Webster, David L. and Robert C. Whitten (1973) Which Electromagnetic Equations Apply in Rotating Coordinates? *Astrophysics and Space Science*, 24, 323-333.

Yildiz, Asim, and C.H. Tang (1966) Electromagnetic Cavity Resonances in Accelerated Systems. *Physical Review*, 146, 947—954.